THEORY OF DEFORMATION OF POROUS SOIL

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The solid phase of soil consists of densely packed bound particles which were formed as the result of crystal growth, the cementation of deposits following lengthy filtration processes, and diffusion. The degree of packing of solid particles is related to the history of formation of the soil structure. Expansion as a result of microfracture and unpacking during deformation is the result of mechanical action. We assume that elastic deformation is negligibly small. To describe the deformation of the dense phase of soil under loads which act for a relatively short time and result in negligible creep we use the Mises limit condition and the associated deformation law [1]

$$\sqrt{s_{ij}s_{ij}} + \alpha\sigma_{kk}/3 = k; \tag{0.1}$$

$$e_{ij} = \lambda(s_{ij}/\sqrt{s_{kl}s_{kl}} + \alpha\delta_{ij}/3), \ \lambda \geqslant 0. \tag{0.2}$$

Here s_{ij} is the deviator of the stress tensor σ_{ij}; e_{ij} is the strain rate; k and α are respectively the coefficients of Coulomb cohesion and friction.

Equations (0.1) and (0.2) show that the volume strain is always positive. However, the dilatational dependence for soils has a more complicated form, and therefore in the structure of soil with a dense phase (0.1), (0.2), we consider the random distribution of pores. In this case the volume macrodeformations will depend on the predominance of one of these processes: expansion in the dense phase, or consolidation as a result of the decrease of the volume of pores existing at the beginning of deformation. We define the statistics of the pore distribution by the random function $\varkappa$ which is 1 in the pore regions and 0 at the remaining points of the macrovolume V. The volume concentration of pores c is given.

The diversity of properties and structures of the distribution of soil components requires the consideration of special theories. In the present case the assumptions made are sufficient to limit the applicability of the theory and the mathematical formulation of the problem of finding a relation between the macrostresses and macrostrains.

Suppose forces p_i act on the surface S of the macrovolume V. We use the theorem of the minimum of the dissipation rate [2] as applied to Eqs. (0.1) and (0.2). From the conditions for the minimum of a functional defined in the domain of the solid phase V_2,

$$\int_{V_2} k\sqrt{\varepsilon_{ij}\varepsilon_{ij}}dV - \int_S p_i v_i dS, \tag{0.3}$$

where ϵ_{ij} is the deviator of the tensor e_{ij} in the class of kinematically possible velocity fields v_i satisfying in domain V_2 the relation

$$e_{kk} = \alpha\sqrt{\varepsilon_{ij}\varepsilon_{ij}}, \tag{0.4}$$

follow Eqs. (0.1) and (0.2), the equilibrium equations, and the boundary equations on S and on the surfaces of the pores.

In the macrovolume V we consider statistically uniform fields $\varkappa$, σ_{ij}, and e_{ij}. We choose the volume V large enough, in the limit occupying all the space x_i. Using the hypothesis of ergodicity of the fields, we calculate the average over the volume V, and denote by angle brackets

$$\langle\sigma_{ij}\rangle = \frac{1}{V}\int_V \sigma_{ij}dV.$$

Since the ensemble average (mathematical expectation) is the same as the volume average, it can be shown that the average values of the integrals over the surface S divided by the volume V for a statistically uniform integrand approach zero as $V \to \infty$ as the ratio of the area to the volume S/V.

In the present case the variational formulation of the statistical problem presupposes the absence of a statistically uniform field of fluctuations v_i' for given $\langle e_{ij}\rangle$, which under condition (0.4) correspond to the minimum of the functional

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$$D = \frac{1}{V} \int_{V_2} k \sqrt{\varepsilon_{ij}\varepsilon_{ij}} dV. \tag{0.5}$$

In this formulation there is an analogy with the corresponding plasticity theory problem [3]. The relation between the stresses $\langle \sigma_{ij} \rangle$ and the macrostrain rates $\langle e_{ij} \rangle$ is determined by the relations

$$\langle \sigma_{ij} \rangle = \partial D / \partial \langle e_{ij} \rangle, \tag{0.6}$$

which also are a consequence of the minimality of the functional (0.3).

The subscripts 1 and 2 following the angle brackets denote averages over the pore regions V_1 and the matrix V_2, and a prime denotes fluctuations relative to the averages over the volume V. It is meaningful to consider the average $\langle e_{ij} \rangle_1$, since it follows from Gauss' theorem

$$\langle e_{ij} \rangle_1 = \frac{1}{2V_1} \int_{S_1} (v_i n_j + v_j n_i)\, dS$$

that the value of $\langle e_{ij} \rangle_1$ is uniquely determined by the velocity of material particles on the surface of the pores S_1 (n_i is the normal to the surface S_1). In particular, the following relations hold:

$$\langle e_{ij} \rangle_1 = \langle e_{ij} \rangle + \langle \varkappa' e'_{ij} \rangle / c, \quad \langle e_{ij} \rangle = c \langle e_{ij} \rangle_1 + (1 - c) \langle e_{ij} \rangle_2. \tag{0.7}$$

The dissipation function D depends on the volume concentration through $c = \langle \varkappa \rangle$ which is a parameter of the history of deformation with the initial value c_0. Taking account of the change in the volumes V_1 and V in the deformation process, we obtain for the rate of change of the concentration $c = V_1/V$ the equation

$$dc/dt = c(\langle e_{kk} \rangle_1 - \langle e_{kk} \rangle). \tag{0.8}$$

1. From the integral inequalities there follow the estimates

$$\begin{aligned} \langle \sqrt{\varepsilon_{ij}\varepsilon_{ij}} \rangle_2 \leqslant \sqrt{\langle \varepsilon_{ij}\varepsilon_{ij} \rangle_2} = \sqrt{(\langle \varepsilon_{ij}\varepsilon_{ij} \rangle - c \langle \varepsilon_{ij}\varepsilon_{ij} \rangle_1)/(1-c)} \leqslant \\ \leqslant \sqrt{(\langle \varepsilon_{ij}\varepsilon_{ij} \rangle - c \langle \varepsilon_{ij} \rangle_1 \langle \varepsilon_{ij} \rangle_1)/(1-c)}. \end{aligned} \tag{1.1}$$

Independently of the inequalities (1.1) we make some estimates of the dissipation function of a macromedium $D(\langle e_{ij} \rangle)$ for arbitrary fixed values of $\langle e_{ij} \rangle$. The minimum value of the functional (0.5) for a constant concentration c depends on the geometrical structure of the region occupied by the solid phase, and varies from zero to a certain finite maximum value. It is clear that the dissipation function of the macrovolume is equal to zero if the solid particles are not bound, i.e., if they are suspended.

We contrast this structure for equal concentrations c with a certain optimally bound region of distribution of the solid phase which corresponds to the maximum value of the dissipation function $D^*(\langle e_{ij} \rangle)$.

We assume that there is an optimal structure of material for which the minimum value of the upper bound (1.1) of the functional (0.5) coincides with D*. Then for the assumed structure

$$D = \min [k\sqrt{1-c}\sqrt{\langle \varepsilon_{ij}\varepsilon_{ij} \rangle - c \langle \varepsilon_{ij} \rangle_1 \langle \varepsilon_{ij} \rangle_1}] \tag{1.2}$$

under condition (0.4).

Similar estimates are obtained in the theory of elastic composites in which a spherical form of inclusions or pores corresponds to the optimum structure of the material [4, 5]. We note that approximation (1.2) indirectly takes account of the binding of the solid phase region for which quantitative representations, for example for branched inclusions, interpenetrating structures, do not exist so far.

We minimize the functional (1.2) for arbitrarily specified statistically uniform fields e_{kk}. Varying with respect to the fluctuations v'_i, and taking account of Eqs. (0.7) for $\langle e_{ij} \rangle_1$, we obtain

$$\frac{1}{2} v'_{i,j} - \langle \varepsilon_{ij} \rangle_1 \varkappa'_{,j} + f'_{,i} = 0,$$

where f′ is the Lagrangian multiplier for the condition $v'_{i,i} = e'_{kk}$.

Assuming that $\varkappa'$ and e'_{kk} are statistically isotropic functions, we follow [3] and find

$$\langle \varkappa' \varepsilon'_{ij} \rangle = \frac{2}{5} c (1 - c) \langle \varepsilon_{ij} \rangle_1. \tag{1.3}$$

$$\langle \varepsilon'_{ij}\varepsilon'_{ij}\rangle = \frac{2}{5}c(1-c)\langle\varepsilon_{ij}\rangle_1\langle\varepsilon_{ij}\rangle_1 + \frac{2}{3}\langle(e'_{kk})^2\rangle.$$

Equations (0.7) and (1.3) lead to the relation

$$\langle\varepsilon_{ij}\rangle_1 = \langle\varepsilon_{ij}\rangle/[1-(2/5)(1-c)], \tag{1.4}$$

and Eq. (1.2) takes the form

$$D = \min\left[k\sqrt{1-c}\sqrt{a\langle\varepsilon_{ij}\rangle\langle\varepsilon_{ij}\rangle + \frac{2}{3}\langle(e'_{kk})^2\rangle}\right]; \tag{1.5}$$

$$a = 1 - c/[1-(2/5)(1-c)]; \tag{1.6}$$

here minimization is performed with respect to e'_{kk} under condition (0.4).

In domains V_1 and V_2 we take account of the values of e_{kk} with an accuracy to within that of the averages $\langle e_{kk}\rangle_1$ and $\langle e_{kk}\rangle_2$. We approximate the value of $\langle e_{kk}\rangle_2$ calculated from condition (0.4) by the relation for the averages over V_2. Thus,

$$e_{kk} = \langle e_{kk}\rangle_1\varkappa + \langle e_{kk}\rangle_2(1-\varkappa), \qquad \langle e_{kk}\rangle_2 = \alpha\sqrt{\langle\varepsilon_{ij}\rangle_2\langle\varepsilon_{ij}\rangle_2}. \tag{1.7}$$

By using Eqs. (0.7) and (1.4), and taking account of (1.7) we find that

$$\langle(e'_{kk})^2\rangle = \frac{1-c}{c}\left[\langle e_{kk}\rangle - \alpha a\sqrt{\langle\varepsilon_{ij}\rangle\langle\varepsilon_{ij}\rangle}/(1-c)\right]^2.$$

The formulas are less cumbersome if Eq. (1.6) is replaced by $a = 1 - c$; the difference between the two is shown in Fig. 1.

After substitution of the values of $\langle(e'_{kk})^2\rangle$ the dissipation function (1.5) becomes

$$D = k(1-c)\sqrt{\langle\varepsilon_{ij}\rangle\langle\varepsilon_{ij}\rangle + \frac{2}{3}c^{-1}\left(\langle e_{kk}\rangle - \alpha\sqrt{\langle\varepsilon_{ij}\rangle\langle\varepsilon_{ij}\rangle}\right)^2}. \tag{1.8}$$

In the deformation process there is a change in the concentration c determined by Eq. (0.8). Substitution of Eqs. (0.7) and (1.7) into (0.8) gives

$$dc/dt = (1-c)(\langle e_{kk}\rangle - \alpha\sqrt{\langle\varepsilon_{ij}\rangle\langle\varepsilon_{ij}\rangle}). \tag{1.9}$$

The dissipation function (1.8) and Eq. (0.6) determine the limit equilibrium condition of the soil [6]

$$\sqrt{\langle s_{ij}\rangle\langle s_{ij}\rangle} + \alpha\langle\sigma_{kk}\rangle/3 - \sqrt{k^2(1-c)^2 - c\langle\sigma_{kk}\rangle^2/6} = 0 \tag{1.10}$$

and the associated law of deformation

$$\langle\varepsilon_{ij}\rangle = \lambda\langle s_{ij}\rangle/\sqrt{\langle s_{kl}\rangle\langle s_{kl}\rangle},\ \lambda \geqslant 0, \qquad \langle e_{kk}\rangle = \lambda(\alpha + (1/2)c\langle\sigma_{kk}\rangle/\sqrt{k^2(1-c)^2 - c\langle\sigma_{kk}\rangle^2/6}). \tag{1.11}$$

In going from Eq. (1.8) for the dissipation function to Eqs. (1.10) and (1.11) the following features are revealed.

Condition (1.10) holds under the restrictions

$$\sigma_1 \leqslant \langle\sigma_{kk}\rangle/3 \leqslant \sigma_2,\ \sigma_1 = -k\frac{1-c}{\sqrt{c}}\sqrt{\frac{2}{3}},\quad \sigma_2 = \alpha^{-1}\sqrt{k^2(1-c)^2 - \frac{3}{2}c\sigma_2^2}.$$

For values of the hydrostatic pressure σ_2 from (1.10) it follows that $s_{ij} = 0$, and in Eqs. (1.11) the values of $\langle e_{ij}\rangle$ are indeterminate, corresponding to a conical point on the limit equilibrium surface. One should expect this singularity, since the conical point is determined as far back as the initial relations (0.1), (0.2).

For pressures σ_1 it follows from (1.11) that $\langle\varepsilon_{ij}\rangle/\langle e_{kk}\rangle = 0$ independently of the values of s_{ij} within certain limits. Hence we obtain $\langle\varepsilon_{ij}\rangle = 0$ and $\langle e_{kk}\rangle \neq 0$. This case corresponds to a law of deformation associated with the condition $\langle\sigma_{kk}\rangle = 3\sigma_1$. It can be shown that in $\langle\sigma_{ij}\rangle$ space the hyperplane $\langle\sigma_{kk}\rangle = 3\sigma_1$ joins smoothly with the hypersurface (1.11).

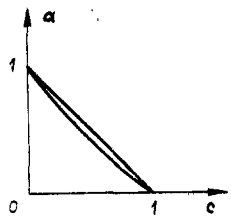

Fig. 1

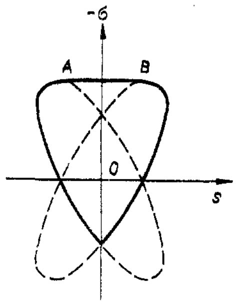

Fig. 2

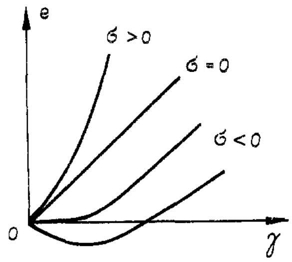

Fig. 3

These features become clear if we represent the surface D = const in e_{ij} space; then from condition (0.7) it follows that the conical point on the surface D = const corresponds to the flat part of the limit equilibrium surface, and the flat part to the conical point in σ_{ij} space.

In order to take account of the consolidation of soils and to preserve the associated law of deformation, phenomenological theories close the limit equilibrium surface by a flat base corresponding to the value of σ_{kk} at which the consolidation process begins [6, 7]. The present work shows that this assumption is not introduced arbitrarily.

2. Let us investigate some properties of soils determined by Eqs. (1.9)-(1.11). We consider the effect of hydrostatic pressure σ and the tangential stress s. Condition (1.10) takes the form

$$|s| + \alpha\sigma - \sqrt{k^2(1-c)^2 - 3c\sigma^2/2} = 0. \tag{2.1}$$

In the σ, s plane Eq. (2.1) describes a domain bounded by segments of elliptical curves (Fig. 2) and a rectilinear portion AB corresponding to the value $\sigma = \sigma_1$. On the dashed lines the quantities in Eq. (2.1) become imaginary.

Experimentally [8] determined domains of limit equilibrium have shapes similar to those shown in Fig. 2.

The deformation properties of soil are important in calculating the penetration of landing gear. The depth of a rut depends on consolidation processes under the surface of a wheel and expansion conditions on the shoulders of the rut. In particular, the prediction of conditions for the landing of aircraft on natural soils is linked with studies of dilatational relations, which in a number of cases have a complicated form [9, 10]. The dilatational relations must take account of the change of the limit condition (2.1) in the deformation process with the use of Eq. (1.9).

We investigate the change in volume strain e for a monotonic increase in the shear strain γ at constant pressure σ. It follows from Eq. (1.11) that

$$de/d\gamma = \alpha + (3/2)c\sigma/\sqrt{k^2(1-c)^2 - 3c\sigma^2/2}, \tag{2.2}$$

and Eq. (1.9) takes the form

$$dc/d\gamma = (1-c)(de/d\gamma - \alpha). \tag{2.3}$$

Since the value of σ is fixed, Eq. (2.1) determines the tangential stress s. Equations (2.2) and (2.3) can be integrated for a given initial deformation of the porosity c_0. We assume that the concentration of pores is small, and take account only of linear terms in it; then after integration of Eqs. (2.2) and (2.3) we obtain

$$e = \alpha\gamma + c_0\left(e^{\frac{3}{2}\frac{\sigma}{k}\gamma} - 1\right). \tag{2.4}$$

Figure 3 shows the dilatational curves (2.4) whose shape is characteristic for sandy soils [10, 11]. The effect of the transformation of consolidation into expansion processes for $\sigma < 0$ has been observed experimentally [10]. The initial relations (0.1) and (0.2) do not permit a description of steady-state processes with variable volume strains.

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